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Prüfer rank of G and relation with the Prüfer ranks of $A_1, A_2,\ldots, A_n$

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ABSTRACT: A group G has finite Prüfer rank r=r(G) if every finitly generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this properly. Clearly subgroups and homomorphic images of groups with finite Prüfer rank also have finite Prüfer rank.
If the locally soluble group $G=A_1\ldots A_n$, with finite Prüfer rank is the product of n subgroups $A_1, A_2,\ldots, A_n$, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of $A_1, A_2,\ldots, A_{n-1}$ and A_n.

***Keywords*: :** Prüfer rank, locally soluble group, product of n subgroups, finitly generated.

INTRODUCTION

In 1955 N.Itô (see (7)) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (see(21)) and L.Redei (1950)(see (22)) considered products of cyclic groups, and around 1965 O.H.Kegel (See (30) & (31)) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see (19)) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See (20)&(1)). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See (20))

In 1940 G. Zappa(see (24)) and in 1950 J.Szip (see (23)) studied bout products of groups concerned finite groups. In 1961 O.H.Kegel (See (8)) and in 1958 H.Wielandt (see (10)) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups .
For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see (1), (2),(3),(4) and (6)) , O.H.Kegel (see (8)), J.C.Lennox (see (12)) , D.J.S. Robinson(see (9) and (15)), J.E. Roseblade(see (13)), and D.I.Zaitsev(see (11) and (18)).

Now, in this paper, we show that if the locally soluble group $G=A_1\ldots A_n$, with finite Prüfer rank is the product of n subgroups $A_1, A_2,\ldots, A_n$, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of $A_1, A_2,\ldots, A_{n-1}$ and B.

2. Priliminaries : (elementary properties and theorems.)

In this section we study the elementary Lemma and theorems, whose using in section 3 and prove of main theorem.

2.1. Lemma:

Let the group G=AB be the product of two subgroups A and B. If A_0 and B_0 are subgroups of finite index of A and B, respectively, then the subgroup H=<A_0, B_0> has index at most mn in G, where $|A:A_0|$=m and $|B:B_0|$=n.

Proof :

Let $\{a_1,\ldots,a_m\}$ be a left transversal of A_0 in A and $\{b_1,\ldots,b_n\}$ a right transversal of B_0 in B. Then.

G=AB= $\bigcup_{i,j} a_i A_0 B_0 b_i = \bigcup_{i,j} (a_i H a_i^{-1}) a_i b_i$

is the union of finitely many right cosets of conjugates of H. It follows from Lemma 2.1 that H has finite H has finite index in G. To obtain the required bound for |G:H|, it is clearly enough to consider the finite factor group G/H_G, where H_G is the core of H in G. Consequently we may suppose that G is finite. Then.

$$|G| = \frac{|A|.|B|}{|A \cap B|} \leq \frac{|A|.|B|}{|A_0 \cap B_0|} = \frac{|A_0|.|B_0|}{|A_0 \cap B_0|} mn \leq |H| mn,$$ And so $|G:H| \leq mn.$

2.2. Lemma:

Let the group G=AB be the product of two subgroups A and B.

(i) If A and B satisfy the maximal condition on subgroups, then G satisfies the maximal condition on normal subgroups.

(ii) If A and B satisfy the minimal condition on subgroups, then G satisfies the minimal condition on normal subgroups.

Proof:

See to(1).

2.3. Theorem :

(See (7)) Let the group G=AB be the product of two abelian subgroups A and B. Then G is metabelian.

Proof :

Let a,a_1 be elements of A and b, b_1 elements of B. Write $b^{a_1} = a_2 b_2$ *and* $a^{b_1} = b_3 a_3$, *whit* a_2, a_3 *in* A *and* b_2, b_3 *in* $B.$ Then

$$[a,b]^{b_1 a_1} = [a, b^{a_1}]^{b_1} = [a, b_2]^{b_1} = [a^{b_1}, b_2] = [a_3, b_2]$$

and

$$[a,b]^{b_1 a_1} = [a^{b_1}, b]^{a_1} = [a_3, b]^{a_1} = [a_3, b^{a_1}] = [a_3, b_2].$$

This proves that the commutators (a,b) and (a_1,b_1) commute. Since the factor group G/(A,B) is abelian, it follows that $G' = [a,b],$ and hence G' is abelian.

2.4. Lemma:

Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized . Moreover, every term of the upper central series of G is factorized.

Proof:

Since S is factorized, we have that S=$(A \cap S)(B \cap S).$ Let x=ab be an element of S, where a is in $A \cap S$ and b is in $B \cap S$ .If c=$a_1 b_1$ is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1, x] = [a_1, ab] = [a_1, b] = [c b_1^{-1}, b] = [c, b]^{b_1^{-1}} = 1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of (4). In particular, the center of G is factorized. It follows from Lemma 1.1.2 of (4) that also every term of the upper central series of G is factorized.

2.5.. Lemma: (See (7)) Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof :

Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.11 have Z(G)=(A $\cap Z(G))(B \cap Z(G)) = 1.$ The centralizer $C = C_G(A \cap C_G(G')) \, contains AG',$ and so is normal in G. Since $B \cap (AZ(C)) \le Z(G) = 1,$ it follows that $AZ(C) = A(B \cap AZ(C)) = A.$ This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.9, we have $A \cap G' \le A \cap C_G(G') \le Z(C) = 1.$

Similarly $B \cap G' \le B \cap C_G(G') \le Z(C) = 1.$ The factorizer $X = X(G')$ has the triple factorization $X = A^* B^* = A^* G' = B^* G'$, Where $A^* = A \cap BG'$ *and* $B^* = B \cap AG'.$ Thus X is nilpotent by Corollary 2.8, so that

$Z(X) = (A \cap Z(X))(B \cap Z(X))$

is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G'] \le N \cap G' \le A \cap G' = 1.$ Therefore we obtain the contradiction $N \le A \cap G_G(G') = 1.$

2.6. Theorem(See (8)&(10)):

If the finite group G=AB is the product of two nilpotent subgroups A and B, then G is soluble.

Proof:

See (4) ,(Theorem 2.4.3).

2.7. Lemma :

Let A and B be subgroups of a group G, and let A_1 and B_1 be subgroups of A and B, respectively, such that $|A : A_1| \le m$ and $|B : B_1| \le n.$ Then $|A \cap B : A_1 \cap B_1| \le mn.$

Proof :

To each left coset $x(A_1 \cap B_1)$ of $A_1 \cap B_1$ in $A \cap B$ assign the pair of left cosets $(xA_1, xB_1).$ Clearly this defines an injective map from the set of left cosets of $A_1 \cap B_1$ in $A \cap B$ into the cartesian product of the set of left cosets of A_1 in A and the set of left cosets of B_1 in B. The lemma is proved.

2.8.. Lemma(See (11)):

Let the finitely generated group G=AB=AK=BK be the product of two ablian-by-finite subgroups A and B and an abelian normal subgroup K of G. Then G is nilpotent-by-finite.

Proof:

Let A_1 and B_1 be abelian subgroups of finite index of A and B, respectively, and let n be a positive integer such that $|A:A_1| \le n$ AND $|B:B_1| \le n$. Since G is finitely generated, it has only finitely many subgroups of each finite index, and hence the intersection H of all subgroups of G with index at most n^4 also has finite index in G. In particular H is finitely generated.

Consider a finite homomorphic image H/N of H. Then N has finite index in G, and hence also its core N_G has finite index in G. Let $p_1,\ldots,p_t$ be the prime divisors of the order of the finite abelian group $K/(K \cap N_G).$ For each $j \le t,$ let $K_j/(K \cap N_G)$ be the $p'_j - component$ of $K/(K \cap N_G).$ Clearly each K, is normal in G and

$\bigcap_{j=1}^{t} K_j = K \cap N_G$. The factor group $\overline{G} = G/K$, has the triple factorization $\overline{G} = \overline{A}\overline{B} = \overline{A}\overline{K} = \overline{B}\overline{K}$,where $\overline{K}$ is a finite normal $p_j - subgroup$ of $\overline{G}$. Clearly

$$|\overline{G} : \overline{A} \cap \overline{B}| = |\overline{G} : \overline{A}|.|\overline{A} : \overline{A} \cap \overline{B}| = |\overline{G} : \overline{A}|.|\overline{G} : \overline{B}|$$
$$= |\overline{K} : \overline{A} \cap \overline{K}|.|\overline{K} : \overline{B} \cap \overline{K}| = p_j^k$$

for some non-negative integer k. On the other hand, $|\overline{A} \cap \overline{B} : \overline{A}_1 \cap \overline{B}_1| \leq n^2$ by Lemma 2.16, so that $|\overline{G} : \overline{A}_1 \cap \overline{B}_1| \leq p_j^k n^2$. As $\overline{A}_1$ and $\overline{B}_1$ are abelian, the intersection $\overline{A}_1 \cap \overline{B}_1$ is contained in the centre of $\langle \overline{A}_1, \overline{B}_1 \rangle$, and the factor group $\langle \overline{A}_1, \overline{B}_1 \rangle / (\overline{A}_1 \cap \overline{B}_1)$ has order at most $p_j^k n^2$. Let $\overline{P} / (\overline{A}_1 \cap \overline{B}_1)$ be a Sylow p_j-subgroup of $\langle \overline{A}_1, \overline{B}_1 \rangle / (\overline{A}_1 \cap \overline{B}_1)$. Then $|\langle \overline{A}_1, \overline{B}_1 \rangle : \overline{P}| \leq n^2$, and since $|\overline{G} : \langle \overline{A}_1, \overline{B}_1 \rangle| \leq n^2$ by Lemma 2.2, we obtain $|\overline{G} : \overline{P}| \leq n^4$. Therefore HK_j/K_j is contained in $\overline{P}$ .As an extension of the central subgroup $\overline{A}_1 \cap \overline{B}_1$ by a finite p_j-group, $\overline{P}$ is nilpotent, so that $H/(H \cap K_j) \simeq HK_j / K_j$ is also nilpotent for each j. Hence. $H \Big/ \left(\bigcap_{j=1}^{t} (H \cap K_j) \right) = H/(K \cap N_G)$ is nilpotent. We have shown that each finite homomorphic image of H is nilpotent . As K is abelian, H is soluble, and hence even nilpotent (Robinson 1972,Part 2,Theorem 10.51).Therefore G is nilpotent-by-finite.

2.9.Lemma:

(See (13)) If N is a maximal abelian normal dubgroup of a finite p-group G, then $r(G) \leq \frac{1}{2} r(N)(5\, r(N) + 1)$.

Proof :

Since $C_G(N) = N$, the factor group G/N is isomorphic with a p-group of automorphism of N. Thus G/N has perüfer rank at most $\frac{1}{2} r(N)(5r(N) - 1)$ (See (15), part2, lemma 7.44), and hence $r(G) \leq \frac{1}{2} r(N)(5r(N) + 1)$.

3. Main Theorem:

In this section by using of sections 1 and 2, we prove the following main theorem.

3.1 . Theorem:

If the locally soluble group $G=A_1 \ldots A_n$, with finite Prüfer rank is the product of n subgroups $A_1, A_2, \ldots, A_n$, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of $A_1, A_2, \ldots, A_{n-1}$ and B.

Proof :

The proof by induction on n. First, let G be a finite p-group for some prime p. If N is a maximal abelian normal subgroup of G, by Lemma 2.9 we have $r(G) \leq \frac{1}{2} r(N)(5r(N)+1)$. Hence it is enough to prove that r=r(N) is bounded by a function of the maximum s of r(A) and r(B). The socle S of N is an elementary abelian group of order p' . Clearly it is sufficient to prove the theorem for the factorizer X(S) of S. Therefore we may suppose that the group G has a triple factorization G=AB=AK=BK, where K is an elementary abelian normal subgroup of G of order p' .

Let e be the least positive integer such that A^{p^e} is contained in B. By Lemma 4.3.3 of (4), we have $|A:A\bigcap B|\leq|A:A^{p^e}|\leq p^{eg(s)-s^2}$ Where $g(s)=\frac{1}{2}s(3s+1).$ *Since*

$$|G|=\frac{|A|.|B|}{|A\bigcap B|}=\frac{|B|.|K|}{|B\bigcap K|},$$

It follows that $|K|=|A:A\bigcap B|.|B\bigcap K|\leq p^{eg(s)-s^2}p^s=p^{eg(s)-s^2+s}$. Hence $r\leq eg(s)-s^2+s\leq eg(s).$ Therefore it is enough to show that $e\leq g(s)+3.$ Therefore it is enough to show that $e\leq g(s)+3.$

Clearly we may suppose that e>1. Let a be an element of A such that $a^{p^{e-1}}$ is not in B, and write $a^{p^{e-1}}$ =xb, with x in K and b in B. Then $[x,a^{p^{e-2}}]\neq 1,$ because otherwise

$$b^p=(x^{-1}a^{p^{e-2}})^p=x^{-p}a^{p^{e-1}}=a^{p^{e-1}},$$

contrary to the choice of a. As K has exponent p, it follows from the usual commutator laws that .

$$[x,a^{p^{e-2}}]=\prod_{i=1}^{p^{e-2}}[x,{}_ia]^{(p^{e_i-2})}=[x,p^e.2a].$$

Thus $\underset{\leftarrow p^{e-2}\rightarrow}{[K,G,\ldots,G]}\neq 1,$ and so |K|> $p^{p^{e-2}}$ since G isa finite p-group. Therefore $p^{p-2}<r\leq eg(s).$ If $e\geq g(s)+4$, then $p^{e-2}\geq 2^{e-2}>(e+1)(e-4)\geq(e+1)g(s)>eg(s).$.

This contradiction shows that $e\leq g(s)+3.$

Suppose now that G=AB is an arbitrary finite soluble group. For each prime p, by Corollary 2.7 there exist Sylow p-subgroups A_p of A and B_p of B such that $G_p=A_pB_p$ is a Sylow p-subgroup of G. As was shown above, r(G_p) is bounded by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by at most f(s) elements. Application of Theorem 4.2.1 of (4) yields that every subgroup of G can be generated by at most f(s)+1 elements, and hence the Prüfer rank of G is bounded by f(s)+1. This proves the theorem is the finite case.

Let G=AB be an arbitrary locally soluble group with finite Prüfer rank. If N is a finite normal subgroup of G, and X=X(N) is its factorizer, then the index $|X:A\bigcap B|$ *is finiteby Lemma 1.1.5.Let Y be the core of* $A\bigcap B$ *in* X. Since the factorized group X/Y is finite, it follows from the first part of the proof that thePrüfer rank of X/Y is bounded by a function of the Prüfer ranks of A and B. As $r(N)\leq r(X)\leq r(Y)+r(X/Y)\leq r(A)+r(X/Y)$ (e.g.see Robinson 1972, Part 1,Lemma 1.44) we obtain that there exists a function h such that $r(N)\leq h(r(A),r(B))=k,$ for every finite normal subgroup N of G. Clearly the same holds for every finite normal section of G.

Let T be the maximum periodic normal subgroup of G. If p is a prime, the group $\bar{T}=T/O_{p'}(T)$ is Chernikov by Lemma 3.2.5 of (4) (See also (16)). Let $\bar{J}$ be the finite residual of $\bar{T}$, and $\bar{S}$ the socle of $\bar{J}$. Since $\bar{S}$ and $\bar{T}/\bar{J}$ are finite, it follows that $r(\bar{T})\leq r(\bar{J})+r(\bar{T}/\bar{J})=r(\bar{S})+r(\bar{T}/\bar{J})\leq 2k.$

As the Sylow p-subgroups of T can be embedded in $\overline{T}$, they have Prüfer rank at most 2k. Application of Theorem 4.2.1 of (4) (See also (14)). yields that every finite subgroup of T can be generated by atmost 2k+1 elements. Hence r(T) $\leq 2k+1.$

The group G/T is soluble (See(15)), Part 2, Lemma 10.39), and so the setoff primes $\pi(G/T)$ is finite by Lemma 4.1.5 of (5)(See also (15)). It follows from Lemma 4.1.4 of (4) (See also (15)) that there exists in G a normal series of finite length $T \leq G_1 \leq G_2 \leq G,$ where G_1/T is torsion-free nilpotent, G_2/G_1 is torsion-free abelian, and G/G_2 is finite. Therefore

$$
\begin{aligned}
r(G) &\leq r(T) + r(G_1/T) + r(G_2/G_1) + r(G/G_2) \\
&\leq r(T) + r_0(G) + r(G/G_2) \\
&\leq r_0(G) + 3k + 1.
\end{aligned}
$$

By theorem 4.1.8 of (4) (See also (3)) we have that $r_0(G) \leq r_0(A) + r_0(B).$ Moreover, $r_0(A) \leq r(A)$ and $r_0(B) \leq r(B)$ by Lemma 4.3.4 of (4) (See also (9)). Therefore $r(G) \leq r(A) + r(B) + 3k + 1.$ The theorem is proved.

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